H^p BOUNDS FOR SPECTRAL MULTIPLIERS

ON RIEMMANIAN MANIFOLDS

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- Let $m(\lambda)$ be a bounded measurable function in $\mathbb{R}^n$ and let T_m be the operator defined by

$$\widehat{T_m f}(\lambda) = m(\lambda)\widehat{f}(\lambda).$$

The Mikhlin-Hörmander multiplier theorem (M-H 1960) asserts that if the multiplier $m(\lambda)$ satisfies the condition

$$\sup_{\lambda\in\mathbb{R}^n} |\lambda|^{\alpha}|\partial^{\alpha}m(\lambda)| < \infty,$$

for any multi-index α, with $|\alpha| \leq \left[\frac{n}{2}\right] + 1$, then T_m is bounded on L^p, $1 < p < \infty$ and from L^1 to L^1_w.

Calderón and Torchinsky extending this theorem (C-T 1977) , proved that if the multiplier $m(\lambda)$ satisfies the condition

$$\sup_{\lambda\in\mathbb{R}^n} |\lambda|^{\alpha}|\partial^{\alpha}m(\lambda)| < \infty,$$

for any multi-index α, with $|\alpha| \leq n\left[\left(\frac{1}{p} - \frac{1}{2}\right)\right] + 1$, then T_m is bounded on $H^p, 0 < p \leq 1$.

- There are many generalizations of those theorems. For example on Manifolds (M-H), Discrete groups, Lie groups, Nilpotent groups, Symmetric spaces, Graphs, Stratified groups e.a....

- My generalization (2010) of (C-T) is on the context of Riemannian manifolds

- Let M be a $n-$dimensional, complete, noncompact Riemannian manifold with $C^\infty-$bounded geometry.We denote by $d(.,.)$ the Riemannian distance, by dx the Riemannian measure, by $B(x,r)$ the ball centered at $x \in M$ with radius $r > 0$ and by $V(x,r)$ its volume.

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- We assume that M satisfies the **doubling volume property**, i.e. there is a constant $c > 0$, such that

$$V(x, 2r) \leq cV(x, r), \ \forall \, x \in M, \ r > 0. \tag{0.1}$$

From (0.1) it follows that there exist constants $c, D > 0$, such that

$$\frac{V(x,r)}{V(x,t)} \leq c\left(\frac{r}{t}\right)^{D}, \ \forall x \in M, \ r \geq t > 0. \tag{0.2}$$

- Let us denote by Δ the **Laplace-Beltrami operator** on M and by $p_t(x,y)$, $t > 0$, $x, y \in M$, the **heat kernel** of M, i.e. the fundamental solution of the heat equation $\partial_t u = \Delta u$. We assume that $p_t(x,y)$ satisfies the following **estimates**: there are constants $c, c' > 0$ such that

$$p_t(x,y) \leq c' \frac{e^{-d(x,y)^2/ct}}{V(x,\sqrt{t})}, \tag{0.3}$$

for all $t > 0$ and $x, y \in M$, and there are constants $c_1, c_2 > 0$ and $\gamma \in (0,1)$, such that for all $t > 0$, and $x, y, z \in M$, with $d(y,z) \leq \sqrt{t}$,

$$|p_t(x,y) - p_t(x,z)| \leq \frac{c_1 e^{-c_2 d(x,y)^2/t}}{V(x,\sqrt{t})} \left(\frac{d(y,z)}{\sqrt{t}}\right)^{\gamma}. \tag{0.4}$$

- The Laplace-Beltrami operator Δ on M is a positive and selfadjoint operator on $L^2(M)$. Thus, by the spectral theorem

$$\Delta = \int_0^\infty \lambda dE_\lambda,$$

where dE_λ is the spectral measure on M.

If $m : \mathbb{R} \to \mathbb{R}$ is a bounded Borel function, by the spectral theorem we can define the operator

$$m(\Delta) = \int_0^\infty m(\lambda) dE_\lambda,$$

which is a bounded operator on $L^2(M)$, with $||m(\Delta)||_{2\to 2} \leq ||m||_\infty$. The function m is called a multiplier and the operator $m(\Delta)$, is called a spectral multiplier.

- Let us set,

$$p_0 = \frac{D}{D+\gamma},$$

and

$$A = A(p) = D\left(\frac{1}{p} - \frac{1}{2}\right) + \varepsilon, \ \varepsilon > 0,$$

for all $p \in (p_0, 1]$. Note that in case when $\mathrm{Ric}\,(M) \geq 0, p_0 = \frac{n}{n+1}$.

- Let us denote by $C^A(\mathbb{R})$ the Lipschitz space of order $A > 0$, and by $H^p(M)$ the Hardy space. Finally, let us fix a function $0 \leq \phi \in C^\infty(\mathbb{R})$, with

$$\phi(t) = 1,\ \forall t \in [1,2],\quad \phi(t) = 0,\ t \in \left(\tfrac{1}{2}, 4\right)^c.$$

In the present work we prove the following

- **theorem:** Let M be a Riemannian manifold as above and let $m(\lambda),\ \lambda \in \mathbb{R}$, be a multiplier satisfying

(0.5) $$\sup_{t>0} ||\phi(.)m(t.)||_{C^{A(p)}} < \infty,\quad p \in (p_0, 1]$$

Then the operator $m(\Delta)$ is bounded on H^p.

We note that by interpolation and duality, from Theorem it follows that $m(\Delta)$ is bounded $L^p(M)$, for $1 < p < \infty$, and on $BMO(M)$.

- exg. $\Delta^{i\beta},\ \beta \in \mathbb{R}$.

- If $p \in (p_0, 1]$, we say that a function a is a p-atom, if there is a ball $B(y,r)$ such that

(0.6) $$supp(a) \subseteq B(y,r),\ ||a||_\infty \leq V(y,r)^{-1/p}$$

and $\int_M a(x)dx = 0$. From (0.6) we get that

(0.7) $$||a||_q \leq V(y,r)^{(1/q)-(1/p)},\quad q \geq 1.$$

- We need first to define the **Lipschitz space** $\mathcal{L}_\alpha$, $\alpha > 0$. We say that $f \in \mathcal{L}_\alpha$, if there is a constant $c > 0$ such that for every ball B and $x, y \in B$, we have

(0.8) $$|f(x) - f(y)| \leq c|B|^\alpha.$$

The norm $||f||_{\mathcal{L}_\alpha}$ is defined as the smallest of those constants c and makes $\mathcal{L}_\alpha$,a Banach space.

For $p \in (p_0, 1)$ we set $\alpha = (1/p) - 1$. Then we define H^p as the space of those functionals $f \in \mathcal{L}_\alpha^{'}$ which can be written as $f = \sum_{n=0}^{\infty} \lambda_n a_n$, where $(\lambda_n) \in \ell^p$ and (a_n) is a sequence of p-atoms. We set

$$||f||_{H^p} = \inf\left\{ \left(\sum_{n=0}^{\infty} |\lambda_n|^p \right)^{1/p};\ f = \sum_{n=0}^{\infty} \lambda_n a_n \right\}.$$

We note that the dual H^p is $\mathcal{L}_\alpha$ and that for every $f \in \mathcal{L}_\alpha$, and for every ball B and $y \in B$, we have that

(0.9) $$||f - f(y)||_{L^2(B)} \leq ||f||_{\mathcal{L}_\alpha} |B|^{(1/p)-(1/2)}.$$

- **Strategy of the proof**
  (1) Let p be in $(p_0, 1)$, a be a p-atom supported on $B(y,r)$, $y \in M$, $r > 0$ and $\psi \in C_0^\infty$. By the duality argument it suffices to show that

$$|\langle m(\Delta)a, \psi\rangle| \leq c||a||_{H^p}||\psi||_{\mathcal{L}_\alpha} = c||\psi||_{\mathcal{L}_\alpha},$$

(2) Cancelation property: For every p-atom a, we have

$$\int_M (m(\Delta)a)(x)dx = 0.$$

Then we write

$$\langle m(\Delta)a, \psi\rangle = \langle m(\Delta)a, \psi - \psi(y)\rangle. \tag{0.10}$$

and $\psi - \psi(y) = \psi_1 + \psi_2$, supported on ball $B(y, 4r)$ and on its complement respectively.

We have then

$$\langle m(\Delta)a, \psi\rangle = \langle m(\Delta)a, \psi_1\rangle + \langle m(\Delta)a, \psi_2\rangle. \tag{0.11}$$

(3) By the Cauchy-Schwarz inequality we get that $|\langle m(\Delta)a, \psi_1\rangle| \leq$

$$||m(\Delta)||_{2\to2}||\ ||_2||\psi - \psi(y)||_{L^2(B(y,4r))}.$$

Using (0.7) and (0.9), it follows from the doubling property that

$$|\langle m(\Delta)a, \psi_1\rangle| \leq c||\psi||_{\mathcal{L}_\alpha}.$$

(4) We cut the multiplier on compactly supported terms m_j and

$$|\langle m(\Delta)a, \psi_2\rangle| \leq \sum_{j<N+4} |\langle m_j(\Delta)a, \psi_2\rangle| + \sum_{j\geq N+4} |\langle m_j(\Delta)a, \psi_2\rangle|,$$

where $N \in \mathbb{Z}$ be such that

$$2^{N/2} \leq r < 2^{(N+1)/2}. \tag{0.12}$$

(5) The second sum is estimated simillarly with the case of graphs.

(6) For the first sum because, $B(y, 4r)^c \subseteq \cup_{q\geq N+4} A_q(y)$, where

$$A_q(y) = B(y, 2^{(q+1)/2}) - B(y, 2^{q/2}),$$

we take by the Cauchy- Swartz

$$|\langle m_j(\Delta)a, \psi_2\rangle| \leq$$

$$\sum_{q\geq N+4} ||m_j(\Delta)a||_{L^2(A_q(y))}||\psi_2||_{L^2(A_q(y))},$$

and by Minkowski inequality , $||m_j(\Delta)a||_{L^2(A_q(y))} \leq$

$$||a||_1 \sup_{d(z,y)\leq r} ||K_j(.,z)||_{L^2(A_q(y))}.$$

Where K_j is the kernel of $m_j(\Delta)$.It suffices to estimate the norm

$$||K_j(.,z)||_{L^2(A_q(y))}$$

but this is a consequence of heat kernel's estimates. In fact we have if $j \leq q$,

$$||K_j(.,y)||_{L^2(B(y,2^{q/2})^c)} \leq \frac{c||m_j||_{C^A} 2^{-A(q-j)/2}}{\sqrt{V(y, 2^{j/2})}}. \tag{0.13}$$

Putting all together with the relations (0.7),(0.9), using the doubling volume property and summing over q and j we have that

$$|\langle m(\Delta)a, \psi_2\rangle| \leq c||\psi||_{\mathcal{L}_\alpha} (q.e.d.)$$

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